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Means and Taylor Polynomials

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INTRODUCTION

Let f be a convex, differentiable real-valued function on $(0, \infty)$, and let $T_a(f) \equiv T_a$ denote the tangent line to the graph of f at $(a, f(a))$. Also, let $i(T_a, T_b) = (i_x(T_a, T_b), i_y(T_a, T_b))$ denote the intersection point of T_a and T_b. A simple computation shows that

$$\min(a, b) \leqslant i_x(T_a, T_b) \leqslant \max(a, b) \tag{1}$$

and thus $i_x(T_a, T_b)$ defines a *mean*, which we denote by $M_f(a, b)$ to emphasize the dependence on the given function f. In general a mean is a continuous function of two positive variables which satisfies (1). It is not hard to show that $i_x(T_a, T_b)$ is a continuous function of a and b. It also follows directly from the definition of M_f that

$$M_f(b, a) = M_f(a, b) \tag{2}$$

so that M_f is a symmetric mean. If a mean $m(a, b)$ also satisfies $m(ka, kb) = km(a, b)$ for any $k > 0$, then m is called homogeneous.

In Section 1 we generalize the means M_f in a natural fashion using intersections of *odd* degree Taylor polynomials. More precisely, if P and Q denote the Taylor polynomials of order r to f at a and b, we prove that $P - Q$ has precisely one zero in (a, b), where $f^{(r+1)} > 0$ on $[a, b]$ and r is an odd integer. We denote this zero by $M_f^r(a, b)$, and it follows that M_f^r satisfies (1) and (2), and thus is a symmetric mean. The homogeneous means in this class are the means $M_p^r \equiv M_f^r$ where $f(x) = x^p$. We shall prove that for each r, $M_{r+1}^r(a, b) = (a + b)/2 =$ the arithmetic mean A, $M_{r/2}^r(a, b) = \sqrt{ab} =$ the geometric mean G, and $M_{-1}^r(a, b) = 2ab/(a + b) =$ the harmonic mean H. In addition, for fixed a and b, $M_p^r(a, b)$ is increasing in p, and this gives the well-known inequality $H \leqslant G \leqslant A$. In Section 2 we consider intersections of certain Hermite interpolants other than Taylor

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polynomials, and define a mean similar to M_f^r. In Section 3 we explore some other means involving intersection points of tangent lines, and prove some apparently new inequalities using basic properties of convex functions. Finally in Section 4 we find the first few derivatives of $h(b) = M_f^r(1, b)$ in terms of various derivatives of f, and use this to compare M_f^r with other means.

0. Preliminaries

In this section we state some important facts about Taylor polynomials and divided differences. Given r and $f \in C^r$, we let P_a denote the Taylor polynomial of order r to f at $x = a$. (Order r means P_a and f agree through their first r derivatives at a, so that the degree of P_a may be less than r. Note that P_a depends, of course, on f and r as well as a.) First we define the error

$$E_a = f - P_a. \tag{0.1}$$

We employ two versions of E_a. First, the Lagrange form of the remainder

$$E_a(x) = \frac{f^{(r+1)}(\zeta_a)}{(r+1)!}(x-a)^{r+1}, \qquad \zeta_a \text{ between } a \text{ and } x. \tag{0.2}$$

For the second version we need to use the divided-differences of f, which are defined inductively for distinct x_j, by $f[x_1, ..., x_k] = (f[x_1, ..., x_{k-1}] - f[x_2, ..., x_k])/(x_1 - x_k)$, starting with $f[x_1] = f(x_1)$. It is well-known [IK] that $f[x_1, ..., x_k]$ can be extended in a continuous fashion to identical x_j's, for suitably differentiable f. The second form we use is

$$E_a(x) = f[a, ..., a, x](x-a)^{r+1}, \tag{0.3}$$

where $f[a, ..., a, x]$ denotes the divided difference of order $r+1$, when $r+1$ of the arguments coalesce into one point a.

Lemma 0.1. *Suppose $f^{(r+1)} > 0$ on $[a, b]$, and let $\{x_1, ..., x_{r+1}\}$ and $\{y_1, ..., y_{r+1}\}$ be any two sets of $r+1$ points (not necessarily distinct) in $[a, b]$ with $y_j \leqslant x_j$ for $j = 1, ..., r+1$,* and *$y_j \neq x_j$ for some j. Then $f[x_1, ..., x_{r+1}] > f[y_1, ..., y_{r+1}]$.*

Remark. Note that by the mean value theorem for divided differences [IK], $f[x_1, ..., x_{r+2}] > 0$ for any set of $r+2$ points $\{x_1, ..., x_{r+2}\}$.

Proof. Write

$$\begin{aligned}
&f[x_1, ..., x_{r+1}] - f[y_1, ..., y_{r+1}] \\
&\quad = (f[x_1, x_{r+1}, ..., x_2] - f[x_{r+1}, ..., x_2, y_1]) + (f[x_2, x_{r+1}, ..., x_3, y_1] \\
&\qquad - f[x_{r+1}, ..., x_3, y_1, y_2]) \\
&\qquad + \cdots + (f[x_{r+1}, y_1, y_2, ..., y_r] - f[y_1, ..., y_{r+1}]) \\
&\quad = f[x_1, x_{r+1}, ..., x_2, y_1](x_1 - y_1) \\
&\qquad + f[x_2, x_{r+1}, ..., x_3, y_1, y_2](x_2 - y_2) \\
&\qquad + \cdots + f[x_{r+1}, y_1, ..., y_{r+1}](x_{r+1} - y_{r+1}) > 0
\end{aligned}$$

by the remark, since all the divided differences appearing have order $r+1$.

Remark. While [B, Theorem 6] is very similar to Lemma 0.1, it is not exactly what we need.

Now suppose $P_a(x_0) = P_b(x_0)$ for some x_0, which implies that $f(x_0) - P_a(x_0) = f(x_0) - P_b(x_0)$. Then we have, for fixed $a < b$,

$$M_f^r(a, b) = x_0 \Leftrightarrow \frac{f^{(r+1)}(\zeta_a)}{(r+1)!}(x_0 - a)^{r+1} = \frac{f^{(r+1)}(\zeta_b)}{(r+1)!} x_0 - b)^{r+1}, \tag{0.4}$$

where ζ_a and ζ_b are in the smallest interval containing a, b, and x_0.

Also

$$M_f^r(a, b) = x_0 \Leftrightarrow f[a, ..., a, x_0](x_0 - a)^{r+1} = f[b, ..., b, x_0](x_0 - b)^{r+1}. \tag{0.5}$$

It is also elementary that

$$P_a'(x) \text{ is the Taylor polynomial of order } r-1 \text{ to } f'(x) \text{ at } x = a. \tag{0.6}$$

1. The Means M_f^r

Throughout this section r will always denote an *odd* positive integer. The following lemma is key in proving that $M_f^r(a, b)$ is a well-defined mean.

Lemma 1.1. *Suppose $f^{(r)}$ is continuous and non-vanishing on $[a, b]$, and let P and Q denote the Taylor polynomials of order $r-1$ to f at a and b, respectively. Then $P - Q$ has no zero in (a, b).*

Remark. It actually follows that $P - Q$ cannot vanish at a or b either, though we make no use of that fact.

Proof. Suppose $P(x_0)=Q(x_0)$ for some x_0 in (a,b). By (0.4), with r replaced by $r-1$, we have

$$\frac{f^{(r)}(\zeta_a)}{r!}(x_0-a)^r=\frac{f^{(r)}(\zeta_b)}{r!}(x_0-b)^r \tag{1.1}$$

for some ζ_a, ζ_b in the smallest interval containing a, b, and x_0. But the left and right-hand sides of (1.1) must have opposite signs, since r is odd, and this proves the lemma.

Thus Lemma 1.1 shows that *even*-degree Taylor polynomials are not useful in defining means—at least not in the manner done in this paper.

THEOREM 1.1. *Suppose $f^{(r+1)}$ is continuous and non-vanishing on $[a,b]$, and let P and Q denote the Taylor polynomials of order to f at a and b, respectively. Then $P-Q$ has precisely one zero in (a,b).*

Proof. Assume, without loss of generality, that $f^{(r+1)}>0$ on $[a,b]$. Let $R_f(x)=P(x)-Q(x)$ (we suppress the dependence of P and Q on f). First, if R_f has more than one zero in (a,b), then R_f' vanishes in (a,b) by Rolle's Theorem, which contradicts Lemma 1.1 (applied to f'), since P' and Q' are Taylor polynomials of order $r-1$ to f' at a and b, respectively. Now let

$$T=\{f\in C^{r+1}[a,b]: f^{(r+1)}>0 \text{ on } [a,b]\}$$

and

$$S=\{f\in T: R_f \text{ has precisely one zero in } (a,b)\}.$$

It is easy to show that T is a connected set in the topology τ of uniform convergence of $f, f', ..., f^{(r+1)}$ on $[a,b]$. Since the zeros of a polynomial are continuous functions of its coefficients, if $R_f(x_0)=0$ for some x_0 in (a,b), then for g sufficiently close to f in the topology τ, R_g must vanish near x_0, and hence in (a,b). This follows because $R_f'(x_0)\neq 0$, again by Lemma 1.1. Thus S is an open set. To show that S is closed, suppose $\{f_j\}$ converges to f in the topology τ, f_j in S, f in T. Then R_f must vanish in $[a,b]$. But $R_f(a)\neq 0\neq R_f(b)$ since $E_a(b)\neq 0\neq E_b(a)$, which follows from $f^{(r+1)}\neq 0$. Hence S is also closed, and since T is connected, either $S=T$ or $S=\varnothing$. Theorem 1.1 then follows from our next result, which shows that $S\neq\varnothing$.

THEOREM 1.2. *If $f(x)=x^{r+1}$, then $R_f((a+b)/2)=0$. In other words, the Taylor polynomials of order r to x^{r+1} at a and b always intersect in a point whose x-coordinate is the arithmetic mean of a and b.*

Proof. For fixed $a<b$, suppose $R_f(x_0)=0$ (since R_f is a polynomial of odd degree, it has a real zero). Since $f^{(r+1)}$ is constant, by (0.4) $(x_0-a)^{r+1}=(x_0-b)^{r+1}$, and the theorem follows immediately.

For fixed f and r, by Theorem 1.1, $M_f^r(a,b)=i_x(P_a,P_b)=$ the unique zero of P_a-P_b in (a,b) defines a *mean* in a and b. We note here that $M_f^r(a,b)$ is continuous in a and b. From here on, when we refer to M_f^r we assume f is at least $C^{r+1}(0,\infty)$ with $f^{(r+1)}(x)\neq 0$ for any $x>0$. It is also easy to show that $M_f^r\equiv M_g^r$ whenever $g=cf+p$, c a constant and p a polynomial of degree $\leqslant r$.

We let M_p^r denote M_f^r when $f(x)=x^p$, and p is not equal to a nonnegative integer $\leqslant r$. If $p\in\{0,1,...,r\}$, we define $M_p^r=M_f^r$ where $f(x)=x^p\log x$. This provides a continuous extension of M_p^r to all real numbers p, which follows immediately from the fact that $x^p\log x=\lim_{t\to 0}((x^{t+p}-x^p)/t)$. It also follows immediately that M_p^r is a homogeneous mean. We shall prove shortly that these are the *only* homogeneous means among the class of means M_f^r (at least if $f\in C^{r+2}(0,\infty)$). First we need the following

LEMMA 1.2. *$M_{f(kx)}^r(a,b)=(1/k)\,M_{f(x)}^r(ka,kb)$ for any $k>0$.*

Proof.

$$x_0=M_f^r(ka,kb)\Leftrightarrow\sum_{j=0}^{r}\frac{f^{(j)}(kb)(x_0-kb)^j-f^{(j)}(ka)(x_0-ka)}{j!}=0.$$

Then

$$\sum_{j=0}^{r}\frac{k^j\left(f^{(j)}(kb)\left(\frac{x_0}{k}-b\right)^j-f^{(j)}(ka)\left(\frac{x_0}{k}-a\right)^j\right)}{j!}=0,$$

which says that $M_{f(kx)}^r(a,b)=x_0/k$ since $M_{f(kx)}^r(a,b)$ is the real root in (a,b) of the polynomial

$$\sum_{j=0}^{r}\frac{k^j(f^{(j)}(kb)(x-b)^j-f^{(j)}(ka)(x-a)^j)}{j!}.$$

(Note that $ka<x_0<kb$.)

COROLLARY. *If M_f^r is homogeneous, then $M_{f(kx)}^r(a,b)=M_{f(x)}^r(a,b)$ for all $k,a,b>0$.*

THEOREM 1.3. *Suppose f and g are in $C^{r+2}(0, \infty)$, with $f^{(r+1)}$ and $g^{(r+1)}$ non-zero on $(0, \infty)$. Assume also that $M_f^r(a, b) = M_g^r(a, b)$ for all $0 < a < b$. Then $g(x) = cf(x) + p(x)$ for some constant c and some polynomial p of degree $\leqslant r$.*

Proof. Suppose that $h \equiv f^{(r+1)}/g^{(r+1)}$ is not a constant function on $(0, \infty)$, which then implies that h is strictly monotone on some interval $[a, b]$ since h' is continuous. Let $x_0 = M_f^r(a, b) = M_g^r(a, b)$, and let P_c and Q_c denote the Taylor polynomials of order r at c to f and g, respectively. Then $f(x_0) - P_a(x_0) = f(x_0) - P_b(x_0)$ and $g(x_0) - Q_a(x_0) = g(x_0) - Q_b(x_0)$ implies

$$\frac{f(x_0) - P_a(x_0)}{g(x_0) - Q_a(x_0)} = \frac{f(x_0) - P_b(x_0)}{g(x_0) - Q_b(x_0)}. \tag{1.2}$$

(Note that the denominators in (1.2) cannot vanish since $g^{(r+1)}$ is never 0.) By Abian's generalization of the generalized Mean Value Theorem [A], (1.2) becomes $f^{(r+1)}(\mu_a)/g^{(r+1)}(\mu_a) = f^{(r+1)}(\mu_b)/g^{(r+1)}(\mu_b)$ where $a < \mu_a < x_0 < \mu_b < b$, which contradicts the fact that $f^{(r+1)}/g^{(r+1)}$ is *strictly* monotone on $[a, b]$. Thus $f^{(r+1)}/g^{(r+1)}$ must be constant on $(0, \infty)$, and this proves the theorem.

THEOREM 1.4. *Suppose $f^{(r+2)}$ is continuous on $(0, \infty)$ and that M_f^r is a homogeneous mean. Then $f^{(r+1)}(x) = cx^p$ for some real numbers c and p.*

Proof. By Theorem 1.3 and the Corollary to Lemma 1.2, $k^{r+1}f^{(r+1)}(kx) \equiv c_k f^{(r+1)}(x)$ for each $k > 0$. A standard argument then shows that $f^{(r+1)}(kx) = f^{(r+1)}(k)\, f^{(r+1)}(x)$, and it is well-known that this implies that $f^{(r+1)}(x) = cx^p$.

It is interesting to examine some of the properties of the means M_p^r. First we have

THEOREM 1.5. *Suppose $f^{(r+1)}$ and $g^{(r+1)}$ are continuous and that $f^{(r+1)}/g^{(r+1)}$ is strictly monotonic on $(0, \infty)$. Let $O = \{(a, b) : 0 < a < b\}$. Then $M_f^r(a, b) < M_g^r(a, b)$ on O, or $M_g^r(a, b) < M_f^r(a, b)$ on O.*

Proof. Suppose $M_f^r(a, b) = M_g^r(a, b) \equiv x_0$ for some $0 < a < b$. Arguing as in the proof of Theorem 1.3, we have $f^{(r+1)}(\mu_a)/g^{(r+1)}(\mu_a) = f^{(r+1)}(\mu_b)/g^{r+1)}(\mu_b)$ where $0 < \mu_a < x_0 < \mu_b < b$, which contradicts the fact that $f^{(r+1)}/g^{(r+1)}$ is *strictly* monotone on $[a, b]$. Since $M_f^r(a, b)$ and $M_g^r(a, b)$ are continuous functions on O, and since O is connected, the theorem then follows from the intermediate value theorem.

COROLLARY. *$M_p^r(a, b)$ is increasing in p for each fixed r and $0 < a < b$.*

Proof. Since $M_p^r(a, b)$ is a continuous function of p on the real line, and since $(x^{p_2})^{(r+1)}/(x^{p_1})^{(r+1)}$ is monotonic whenever $p_1 \neq p_2$, by Theorem 1.5, $M_p^r(a, b)$ must be either increasing or decresing in p. The corollary will then follow from the next result.

THEOREM 1.6. *$\lim_{p\to\infty} M_p^r(a, b) = \max\{a, b\}$ and $\lim_{p\to-\infty} M_p^r(a, b) = \min\{a, b\}$, each fixed r.*

Proof. Let $R_p(x) = \sum_{k=0}^r \binom{p}{k}(b^{p-k}(x-b)^k - a^{p-k}(x-a)^k)$, $S_p(x) = R_p(x)/\binom{p}{r}(b^{p-r} - a^{p-r})$, so that $M_p^r(a, b)$ is the root of $S_p(x)$ in (a, b). Since M_p^r is homogeneous and symmetric, it suffices to let $a = 1 < b$. Also, since

$$\lim_{p\to\infty} \frac{\binom{p}{k}}{\binom{p}{r}} = \begin{cases} 0 & \text{if } k < r \\ 1 & \text{if } k = r \end{cases},$$

it follows easily that as $p \to \infty$, S_p converges to $(x-b)^r$ uniformly on compact subsets of the complex plane. It follows that $M_p^r(1, b)$ converges to b as p tends to ∞. The case $p \to -\infty$ follows in a similar fashion.

It is interesting to ask which of the well-known means (e.g., the Minkowski means) appears in the set of means M_p^r. We have already seen (Theorem 1.2) that $M_{r+1}^r(a, b) =$ arithmetic mean. The following two theorems show that both the harmonic and geometric means appear as well for each r.

THEOREM 1.7. *$M_{-1}^r(a, b) = 2ab/(a+b)$. In other words, the Taylor polynomials of order r to $1/x$ at a and b always intersect in a point whose x-coordinate is the harmonic mean of a and b.*

Proof. Consider

$$P_b - P_1 = \sum_{k=0}^{r} (-1)^k [\![b^{-1-k}(x-b)^k - (x-1)^k]\!]. \tag{1.3}$$

It suffices to show that (1.3) is 0 when $x = H(1, b) = 2b/(b+1)$, by the homogeneity of H, and since $H(1, b)$ is between 1 and b. The kth and $(k+1)$st terms in the sum in (1.3), k even, look like

$$\frac{1}{b}\left(\frac{b-1}{b+1}\right)^k - \left(\frac{b-1}{b+1}\right)^k + \frac{1}{b}\left(\frac{b-1}{b+1}\right)^{k+1} + \left(\frac{b-1}{b+1}\right)^{k+1}. \tag{1.4}$$

A little algebra shows that (1.4) is 0, and this completes the proof.

THEOREM 1.8. *$M^r_{r/2}(a, b)=\sqrt{ab}$. In other words, the Taylor polynomials of order r to $x^{r/2}$ at a and b always intersect in a point whose x-coordinate is the geometric mean of a and b.*

Proof. It suffices to show that $M^r_{r/2}(b, 1/b)=1$ for all $b>1$. Now $M^r_{r/2}(b, 1/b)$ is the root, in $(b, 1/b)$, of

$$R(x)=\sum_{k=0}^{r}\binom{r/2}{k}\left(b^{r/2-k}(x-b)^k-\left(\frac{1}{b}\right)^{r/2-k}\left(x-\frac{1}{b}\right)^k\right). \tag{1.5}$$

Hence we must show that $R(1)=0$. A little algebra shows that $R(1)$ equals

$$\begin{aligned} &b^{r/2}\sum_{k=0}^{r}\binom{r/2}{k}\left(\frac{1}{b}-1\right)^k-b^{-r/2}\sum_{k=0}^{r}\binom{r/2}{k}(b-1)^k \\ &\equiv f_r\left(\frac{1}{b}\right)-f_r(b), \end{aligned} \tag{1.6}$$

where f_r is the second summation. It thus suffices to show

$$f_r(b)=f_r\left(\frac{1}{b}\right) \qquad \text{for all} \quad b>1. \tag{1.7}$$

We prove (1.7) by induction on r. First, $f_1(b)=\frac{1}{2}(b^{1/2}+b^{-1/2})$, and so (1.7) holds for $r=1$. Now assume (1.7) holds for r, an odd positive integer. Consider

$$\begin{aligned} f_{r+2}(b)&=b^{-r/2-1}\left(\sum_{k=1}^{r+2}\binom{r/2+1}{k}(b-1)^k+1\right) \\ &=b^{-r/2-1}\left(\sum_{k=1}^{r+2}\binom{r/2}{k}(b-1)^k+\sum_{k=1}^{r+2}\binom{r/2}{k-1}(b-1)^k+1\right) \end{aligned}$$

(we use the identity $\binom{r/2+1}{k}=\binom{r/2}{k}+\binom{r/2}{k-1}$), which equals

$$\begin{aligned} &\frac{1}{b}\left(b^{-r/2}\left(\sum_{k=1}^{r}\binom{r/2}{k}(b-1)^k+1\right)\right. \\ &\qquad +b^{-r/2}\sum_{k=0}^{r}\binom{r/2}{k}(b-1)^{k+1}\Bigg) \\ &\qquad +b^{-(r/2+1)}\left(\binom{r/2}{r+1}(b-1)^{r+2}\right. \\ &\qquad \left.+\binom{r/2}{r+2}(b-1)^{r+2}+\binom{r/2}{r+1}(b-1)^{r+1}\right) \\ &=\frac{1}{b}\left(f_r(b)+(b-1)f_r(b)\right)+w(b)=f_r(b)+w(b), \end{aligned}$$

where

$$w(b)=\frac{(b-1)^{r+1}}{b^{r/2+1}}\left(\binom{r/2}{r+1}+\left(\binom{r/2}{r+1}+\binom{r/2}{r+2}\right)(b-1)\right).$$

Thus we have shown

$$f_{r+2}(b)=f_r(b)+w(b). \tag{1.8}$$

We shall also prove

$$w(b)=w\left(\frac{1}{b}\right) \qquad \text{for all } b. \tag{1.9}$$

Then by the induction hypothesis, (1.8), and (1.9), $f_{r+2}(1/b)=f_r(1/b)+w(1/b)=f_r(b)+w(b)=f_{r+2}(b)$, and hence (1.7) holds for $r+2$. Theorem 1.8 then follows by induction. To prove (1.9), using the easily proven fact that

$$\binom{r/2}{r+1}+\binom{r/2}{r+2}=-\binom{r/2}{r+2},$$

it follows that

$$w(b)=\frac{(b-1)^{r+1}}{b^{r/2+1}}\left(-\binom{r/2}{r+2}b-\binom{r/2}{r+2}\right).$$

Thus we have

$$w\left(\frac{1}{b}\right)=\frac{(1-b)^{r+1}b^{r/2+1}}{b^{r+1}}\left(-\binom{r/2}{r+2}(1/b)-\binom{r/2}{r+2}\right)=w(b),$$

using the fact that $r+1$ is *even*.

Our next result compares M_f^r to the arithmetic mean.

THEOREM 1.9. *Suppose* $f^{(r+1)}f^{(r+2)}>0$ (<0) *on* $(0, \infty)$. *Then* $M_f^r(a, b)$ *is* $>$ $(<)(a+b)/2$.

Proof. Letting $x_0=M_f^r(a, b)$ for fixed $0<a<b$, by (0.5) we have $f[a, ..., a, x_0](x_0-a)^{r+1} = f[b, ..., b, x_0](x_0-b)^{r+1}$. Without loss of generality assume that $f^{(r+2)}$ is positive on $(0, \infty)$. Then by Lemma 0.1, $f[a, ..., a, x_0]<f[b, ..., b, x_0]$, the divided differences there being of order $r+1$. If $f^{(r+1)}$ is positive on $(0, \infty)$, then so are all the $(r+1)$st order divided differences with arguments in $(0, \infty)$, by the mean value theorem

for divided differences. It follows then that $(x_0 - a)^{r+1} > (x_0 - b)^{r+1}$, and thus $x_0 > (a+b)/2$. The proof is similar if $f^{(r+1)}$ is negative on $(0, \infty)$.

It is interesting to analyze what happens to M_f^r as $r \to \infty$. First we prove

THEOREM 1.10. *Suppose* $0 < \inf f^{(r+1)}(x) < \sup f^{(r+1)}(x) < \infty$, *where the* inf *and* sup *are taken over* $a \leqslant x \leqslant b$ *and* $r = 1, 3, 5, \ldots$. *Then* M_f^r *approaches the arithmetic mean as r approaches infinity* (*i.e.*, $\lim_{r \to \infty} M_f^r(a, b) = (a+b)/2$ *for each fixed a and b*).

Proof. By (0.4), for each fixed r, a, and b, we have

$$\frac{f^{(r+1)}(\zeta_b)}{f^{(r+1)}(\zeta_a)} = \frac{(x_0 - a)^{r+1}}{(x_0 - b)^{r+1}}, \tag{1.10}$$

where $x_0 = M_f^r(a, b)$ depends on r. Suppose $M_f^r(a, b)$ has a subsequence which approaches $L \neq (a+b)/2$ as $r \to \infty$. Then the RHS of (1.10) has a subsequence which approaches 0 or ∞. But the LHS of (1.10) cannot approach 0 or ∞ by our assumptions on f. This proves the theorem by contradiction.

Remark. Theorem 1.10 is certainly not true for all f, as Theorem 1.7 shows for, say, $f(x) = 1/x$. In fact it is plausible that

(#) *Conjecture.* $\lim_{r \to \infty} M_p^r(a, b) = 2ab/(a+b) =$ harmonic mean.

If one expands $h(b) = M_p^r(1, b)$ in a series about $b = 1$, assuming h is, say, analytic at 1, then (#) holds for terms through $h''''(1)$. This can easily be checked using Eqs. (4.1)–(4.3) from Section 4. Indeed it follows that $\lim_{r \to \infty} h^{(j)}(1)$ is independent of p for $j = 0, 1, \ldots, 4$. If this were true for all j, then the conjecture would follow from Theorem 1.7.

2. HERMITE INTERPOLANTS

In this section we define new means using intersection points of certain Hermite interpolants to f. The following theorem is similar in spirit to Theorem 1.1, though the proof is somewhat different.

THEOREM 2.1. *Suppose* $f^{(r+1)} > 0$ *on* $[a, b]$, *r an odd positive integer. Let* m_1 *and* m_2 *be positive integers with*

$$m_1 + m_2 = r - 1 \tag{2.1}$$

$$m_1 = m_2 + 2. \tag{2.2}$$

Let P and Q be the unique polynomials of degree $\leqslant r$ satisfying the following Hermite interpolation conditions:

$$P^{(j)}(a)=f^{(j)}(a) \quad \textit{for} \quad j=0, 1, ..., m_1$$

and (2.3)

$$P^{(j)}(b)=f^{(j)}(b) \quad \textit{for} \quad j=0, 1, ..., m_2$$

$$Q^{(j)}(a)=f^{(j)}(a) \quad \textit{for} \quad j=0, 1, ..., m_2$$

and (2.4)

$$Q^{(j)}(b)=f^{(j)}(b) \quad \textit{for} \quad j=0, 1, ..., m_1.$$

Then $P-Q$ has precisely one zero in (a, b).

Proof. By (2.3) and (2.4), $R_f=P-Q$ has zeros of order m_2+1 at both a and b. Hence if $R_f(x_0)=0$, $a<x_0<b$, then R_f has at least $2m_2+3$ zeros, counting multiplicities. Thus R_f cannot have more than one zero in (a, b), counting multiplicities, else R_f has at least $r+1$ zeros, which implies $R_f\equiv 0$, which cannot happen since $m_1\neq m_2$. Thus we have proven

$$R_f(x_0)=0, \quad a<x_0<b, \quad \text{implies that} \quad R_f'(x_0)\neq 0. \tag{2.5}$$

Also R_f has at most one zero in (a, b).

Now as in the proof of Theorem 1.1, let

$$T=\{f\in C^{r+1}[a, b]: f^{(r+1)}>0 \text{ on } [a, b]\}$$

$$S=\{f\in T: R_f \text{ has precisely one zero in } (a, b)\}.$$

By (2.5), S is an open subset of T. To show that S is closed, suppose $f_j\in S$, $\{f_j\}$ converging to f in the topology τ from the proof of Theorem 1.1, with $f\in T$. Letting $R_j=R_{f_j}$, we have $R_j(x_j)=0$, $a<x_j<b$, since $f_j\in S$. Choosing a subsequence if necessary, $x_j\to x_0\in [a, b]$. Letting $R=R_f$, we have that $R_j^{(k)}$ converges uniformly in $[a, b]$ to $R^{(k)}$, $k=0, 1, ..., r+1$. It follows that $R(x_0)=0$. We shall prove that $a<x_0<b$. Suppose, by way of contradiction, that $x_0=b$. Now $R_j^{(k)}(b)=0$ for all j and $k=0, 1, ..., m_2$ by (2.3) and (2.4). Since $0=(R_j(b)-R_j(x_j))/(b-x_j)=R_j'(\zeta_j)$, $x_j<\zeta_j<b$, and $\zeta_j\to b$, it follows that $R'(b)=0$. Next we consider $0=(R_j'(b)-R_j'(\zeta_j))/(b-\zeta_j)=R_j''(v_j)$, $\zeta_j<v_j<b$. Since $v_j\to b$, $R''(b)=0$. Continuing in this fashion we see that $R^{(j)}(b)=0$ for $j=0, 1, ..., m_2+1$. This means that $P^{(j)}(b)=Q^{(j)}(b)$ for $j=0, 1, ..., m_2+1$. Since $m_2+1\leqslant m_1$, by (2.4) this says that $P^{(j)}(b)=f^{(j)}(b)$ for $j=0, 1, ..., m_2+1$. From (2.3), $f-P$ has a total of $m_1+1+m_2+2=r+2$ zeros, which cannot happen since $f^{(r+1)}\neq 0$ in $[a, b]$. Thus $x_0\neq b$. A similar argument yields $x_0\neq a$, and hence $a<x_0<b$, which implies that R vanishes at least once in (a, b). If R has more than one zero in (a, b), by (2.5) this is also true for R_j, which contradicts the

fact that $f_j \in S$. Thus $f \in S$, which implies S is a closed subset of T. $S \neq \varnothing$ by the following theorem, and thus $S = T$. This proves Theorem 2.1.

THEOREM 2.2. *Let $f(x) = x^{r+1}$, and suppose P and Q satisfy* (2.3) *and* (2.4). *Then $P - Q$ has precisely one zero x_0 in (a, b), $x_0 = (a+b)/2$.*

Proof. Just like the proof of Theorem 1.2, using a standard formula for the error in polynomial interpolation [IK].

We let $M_f(m_1, m_2; a, b)$ denote the unique root in (a, b) of $P - Q$ guaranteed by Theorem 2.1. It follows that $M_f(m_1, m_2; a, b)$ is a mean in a and b.

Question. Does Theorem 2.1 hold without (2.2) (assuming $m_1 \neq m_2$ of course)?

We can show that the answer is yes if $f(x) = 1/x$, where the corresponding mean is again the harmonic mean.

EXAMPLE. Let $f(x) = x^5$, $m_1 = 2$, $m_2 = 0$, which implies $r = 3$. If $a = 1$ and $b = 2$, then $M_f(2, 0; 1, 2)$ is precisely 23/15, a root of the polynomial $15x^3 - 68x^2 + 99x - 46$. It is interesting to note that $M_5^3(1, 2)$ is the root in $(1, 2)$ of $30x^3 - 140x^2 + 225x - 124$, which has no rational roots. Thus $M_f(2, 0; 1, 2) \neq M_5^3(1, 2)$ (numerically, $M_f(2, 0; 1, 2) \doteq 1.533333$, while $M_5^3(1, 2) \doteq 1.533945$).

3. $r = 1$, SOME NEW MEANS, AND SOME INEQUALITIES

For $r = 1$, $M_f^r(a, b)$ is just the x-coordinate of the intersection point of T_a and T_b, the tangent lines to f at $(a, f(a))$ and $(b, f(b))$. While much in this section can be generalized to any r, r odd, we found it most interesting to concentrate on the case $r = 1$.

It is not hard to show that Theorem 1.3 follows with only the assumption that f is convex or concave on $(0, \infty)$, and we make that assumption throughout this section. Using M_f for M_f^1, we have

$$M_f(a, b) = \frac{(bf'(b) - f(b)) - (af'(a) - f(a))}{f'(b) - f'(a)}. \tag{3.1}$$

Also, if f' is absolutely continuous, then $M_f(a, b) = (\int_a^b xf''(x)\,dx)/(\int_a^b f''(x)\,dx)$. If $f(x) = x^p$, then $M_f(a, b) \equiv M_p(a, b) = ((p-1)/p)((b^p - a^p)/(b^{p-1} - a^{p-1}))$ $(p \neq 0$ or $1)$. The limiting cases $p = 0$ and 1 correspond to $f(x)=\log x$ and $x\log x$, respectively. In the latter case we get the Logarithmic mean $L(a,b)=(b-a)/(\log b-\log a)$. In fact the means M_p are just a special case of a class of means of Stolarsky [S], given by $u(a,b;\alpha,\beta)=((\beta/\alpha)((b^\alpha-a^\alpha)/(b^\beta-a^\beta)))^{1/(\alpha-\beta)}$, with $\beta=p-1$ and $\alpha=p$. For $r=3$, say, the means M_p^r are *not* a subset of Stolarsky's means. This can be shown using Theorem 4.1 and the second and fourth terms of the series expansion of u about $b=1$, with $a=1$ (see [GM]).

The Means N_f

Suppose now that f is monotone, and that the range of f contains $(0,\infty)$. We then define

$$N_f(a,b)=f(M_f(f^{-1}(a),f^{-1}(b))). \tag{3.2}$$

Geometrically this means the following: Given points a and b on the y-axis, $\exists!$ a' and b' such that $f(a')=a$ and $f(b')=b$. $N_f(a,b)$ is then the value of f at $i_x(T_{a'},T_{b'})=M_f(a',b')$. If f is increasing and $a<b$, then $a=f(a')<f(M_f(a',b'))<f(b')=b$. Thus N_f is a mean. For $f(x)=x^p$, $N_f\equiv N_p$ is given by $((b^q-a^q)/(q(b-a)))^{1/(q-1)}$, where $q=(p-1)/p$. Again this is a special case of Stolarsky's means $u(a,b;\alpha,\beta)$ with $\alpha=q$ and $\beta=1$. In particular $N_{1/x}=A$ and $N_{\sqrt{x}}=G$. It is also interesting to note the following facts about N_f:

1. N_f can be homogeneous even when $f\neq x^p$. For example, $N_{e^x}(a,b)=I(a,b)=(1/e)(b^b/a^a)^{1/(b-a)}$ (identric mean). We do not know precisely when N_f is homogeneous (of course N_p is always homogeneous).

2. The set of means N_p is a subset of the set of means $(f')^{-1}((f(b)-f(a))/(b-a))$ (see [M] for a discussion of these means) when $f(x)=x^q$, $q=(p-1)/p$. In particular, from [M] this implies that N_p is never equal to the harmonic mean (including the limiting cases $p=0,1,\infty,-\infty$). It appears, however, that the set of means N_f is *not* a subset of the set of means $(f')^{-1}((f(b)-f(a))/(b-a))$ in general.

The Means S_f

Again assume that f is monotone, and that the range of f contains $(0,\infty)$. Given points a and b on the y-axis, $\exists!$ a' and b' such that $f(a')=a$ and $f(b')=b$. Let $L\equiv L_{a,b}=$ the secant line through $(a',f(a'))$ and $(b',f(b'))$. We then define

$$S_f(a,b)=L(M_f(a',b'))=\text{value of } L \text{ at } i_x(T_{a'},T_{b'}). \tag{3.3}$$

For $f(x)=x^p$, $S_f(a,b)\equiv S_p(a.b)=a+((b-a)/(b^{1-q}-a^{1-q}))((q(b-a)-a^{1-q}b^q+a)/(b^q-a^q))$, where $q=(p-1)/p$.

For example $S_{x^2}=A$, $S_{\sqrt{x}}=H$, $S_{1/x}=(a^2+b^2)/(a+b)$, and $S_{e^x}=2A-L$, a homogeneous mean. We state some facts about S_p:

1. $\lim_{p\to 1} S_p=L$, $\lim_{p\to 0^+} S_p=\min\{a,b\}$, $\lim_{p\to 0^-} S_p=\max\{a,b\}$, and $\lim_{p\to\infty} S_p=\lim_{p\to-\infty} S_p=S_{e^x}$.

2. S_p is never equal to the geometric mean. For example, $S_p(1,4)=2\Rightarrow p\doteq .794$, while $S_p(1,100)=10\Rightarrow p\doteq .755$.

Inequalities

We now prove some inequalities involving M_f, N_f, and S_f.

THEOREM 3.1. *Suppose f is a monotone, convex function on* $(0,\infty)$, *with the range of f containing* $(0,\infty)$. *Then* $M_{f^{-1}}(a,b)\leqslant N_f(a,b)\leqslant S_f(a,b)$, *with strict inequality if $a\neq b$. If f is concave, then the inequalities are reversed.*

Proof. First, it is easy to show that $i_y(T_{a'},T_{b'})$, the y-coordinate of the intersection point of $T_{a'}$ and $T_{b'}$, is equal to $M_{f^{-1}}(a,b)$, where $f(a')=a$ and $f(b')=b$. It also follows that if f is convex, then $i_y(T_{a'},T_{b'})\leqslant f(i_x(T_{a'},T_{b'}))$, with strict inequality if $a\neq b$. Thus $M_{f^{-1}}(a,b)\leqslant N_f(a,b)$, with strict inequality if $a\neq b$. Also, since the secant line L must lie above the graph of f between $(a',f'(a'))$ and $(b',f(b'))$, $f(i_x(T_{a'},T_{b'}))\leqslant L(i_x(T_{a'},T_{b'}))$, again with strict inequality if $a\neq b$. Thus $N_f(a,b)\leqslant S_f(a,b)$, with strict inequality if $a\neq b$. The proof of Theorem 3.1 when f is concave is similar.

Examples

1. $f(x)=x^2$. Since f is convex we have $M_{\sqrt{x}}(a,b)\leqslant N_{x^2}(a,b)\leqslant S_{x^2}(a,b)$, which says that $\sqrt{ab}\leqslant((\sqrt{a}+\sqrt{b})/2)^2\leqslant(a+b)/2$.

2. $f(x)=1/x$. Then we have $M_{1/x}(a,b)\leqslant N_{1/x}(a,b)\leqslant S_{1/x}(a,b)$, which says that $2ab/(a+b)\leqslant(a+b)/2\leqslant(a^2+b^2)/(a+b)$.

3. $f(x)=e^x$. $M_{\log x}\leqslant N_{e^x}\leqslant S_{e^x}$ says that $ab(\log b-\log a)/(b-a))\leqslant (1/e)(b^b/a^a)^{1/(b-a)}\leqslant a+b-(b-a)/(\log b-\log a)$.

4. $f(x)=\sqrt{x}$. This yields $A\geqslant G\geqslant H$.

4. SERIES EXPANSION FOR M_f^r AND SOME COMPARISONS

Suppose now that f and $h(b)=M_f^r(1,b)$ are both analytic at 1, and let their series expansions be given by $f(x)=\sum_{j=0}^{\infty}a_j(x-1)^j$ and $h(b)-b=\sum_{j=1}^{\infty}c_j(b-1)^j$. Then we have

THEOREM 4.1.

(i) $c_1 = -\frac{1}{2}$

(ii) $c_2 = \frac{a_{r+2}}{4a_{r+1}}$

(iii) $c_3 = \frac{(r+3)\,a_{r+3}a_{r+1} - (r+2)(a_{r+2})^2}{8(a_{r+1})^2}$

(iv) $c_4 = \frac{\left(\begin{array}{c}(r+3)(2r+7)\,a_{r+4}(a_{r+1})^2 - 3(r+3)(2r+3)\\ \times a_{r+3}a_{r+2}a_{r+1} + (4r^2+14r+9)(a_{r+2})^3\end{array}\right)}{48(a_{r+1})^3}.$

The author will supply the details of the proof of Theorem 4.1 upon request.

We now use Theorem 4.1 to compare the means M_p^r, $r > 1$, with the means M_q $(r = 1)$.

THEOREM 4.2. *If* $r > 1$, *then the only means that the sets* $\{M_p^r\}$ *and* $\{M_q\}$ *have in common are* A, G, *and* H.

Proof. Assume that the means M_p^r and M_q are identical for some p and q. Since $M_q(1, b) = ((q-1)/q)(b^{q-1} - 1)/(b^q - 1))$ is analytic at $b = 1$, the same is true for $M_p^r(1, b)$. From Theorem 4.1(ii) to (iv), we have, for $h(b) = M_p^r(1, b)$

$$h''(1) = \frac{(p-r-1)}{2(r+2)} \tag{4.1}$$

$$h'''(1) = \frac{-3(p-r-1)}{4(r+2)} \tag{4.2}$$

$$\begin{aligned} h''''(1) &= (p-r-1)\frac{\left(\begin{array}{c}4(r+2)^2(2r+7)(p-r-2)(p-r-3)\\ -12(r+4)(r+2)(2r+3)(p-r-1)(p-r-2)\end{array}\right)}{8(r+2)^3(r+4)} \\ &\quad + (p-r-1)\left(\frac{4(r+4)(4r^2+14r+9)(p-r-1)^2}{8(r+2)^3(r+4)}\right). \end{aligned} \tag{4.3}$$

Then using (4.1) or (4.2), we have

$$p = \frac{r+2}{3}q + \frac{r-1}{r+2}. \tag{4.4}$$

Denote the right-hand side of (4.3) by $A(p, r)$. Since $M_q(1, b) = M_p^r(1, b)$ for all $b \geqslant 1$, we have $A(p, r) = A(q, 1)$ which implies, from (4.4), that $A(((r+2)/3)q + (r-1)/(r+2), r) = A(q, 1)$. This says

$$R_1(q) = R_2(q) \tag{4.5}$$

for some polynomials R_1 and R_2 of degree $\leqslant 3$ with coefficients that depend on r. Now suppose that $\deg(R_1 - R_2) < 3$. Then since the coefficient of q^3 in $R_1 - R_2$ is 0, this leads, through some simplification, to the equation $66r(r-1) - 136(r-1) + 264(r+1)(r-1) + 812(r-1) = 0$. Since $r \neq 1$, this becomes $33r + 132(r+1) + 338 = 0$, a contradiction since r is positive. Hence $R_1 - R_2$ has degree $= 3$, and thus Eq. (4.5) has at most 3 distinct real roots. But from Theorems 1.2, 1.7, and 1.8, for each r those roots must be $q = 2, -1$, and $1/2$, which of course give the arithmetic, harmonic, and geometric means.

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